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The boundary correction for the Rayleigh–Darcy problem: limitations of the Brinkman equation

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The no-slip condition on rigid boundaries necessitates a correction to the critical value of the Rayleigh–Darcy number for the onset of convection in a horizontal layer of a saturated porous medium uniformly heated from below. It is shown that the use of the Brinkman equation to obtain this correction is not justified, because of the limitations of that equation. These limitations are discussed in detail. An alternative procedure, based on a model in which the porous medium is sandwiched between two fluid layers, and the Beavers–Joseph boundary condition is applied at the interfaces, is described, and an expression for the correction is obtained. It is found that the correction can be of either sign, depending on the relative magnitudes of the parameters involved.

1. Introduction

Rudraiah, Veerappa & Balachandra Rao (1980) have claimed that 'to understand the onset of convection in a porous medium made up of sparse distribution of particles one has to take into account the viscous shear, however small it may be, in addition to the Darcy resistance. In other words, instead of considering only the potential nature of the Darcy equation, one has to consider also the boundary layer type of equation as postulated for the first time by Brinkman [1947*a*] (hereafter called the Brinkman model) for which a rigorous theoretical justification was given later by Tam [1969] and Lundgren [1972].' Their point is that at a rigid boundary the no-slip condition must be applied, and this is inconsistent with the use of the usual Darcy equation. In the present paper we point out the dangers of using the Brinkman equation in the way which Rudraiah *et al.* did so. Later we indicate an alternative way of dealing with the no-slip condition, but first we take a critical look at the Brinkman model.

2. The Brinkman equation in its original context

Brinkman (1947*a*) considered slow steady flow through a porous bed of spherical particles with the porosity sufficiently large for one to take the equation for flow past an individual sphere to be

$$\nabla p = -\frac{\mu}{K}\mathbf{v} + \tilde{\mu}\nabla^2\mathbf{v}, \qquad (2.1)$$

where $\mathbf{v}$ and p are the fluid velocity and pressure, μ is the viscosity of the fluid and $\tilde{\mu}$ is an effective viscosity. For an incompressible fluid, $\mathbf{v}$ satisfies

$$\nabla . \mathbf{v} = 0. \qquad (2.2)$$

Brinckman solved (2.1) and (2.2) subject to the appropriate boundary conditions ($\mathbf{v} = 0$ on the surface of the sphere, and $\mathbf{v} = \mathbf{v}_0$ at large distances from the sphere). He calculated the drag on the sphere to be mD_s, where $D_s = 6\pi\tilde{\mu}v_0 a$ is the Stokes drag on a sphere of radius a, and $m = 1+\lambda a+\frac{1}{3}\lambda^2a^2$, where $\lambda = (\mu/K\tilde{\mu})^{\frac{1}{2}}$. He then identified v_0 with the unidirectional mean filter velocity and equated the total force on the spheres contained in a column of the medium to the Darcy drag on that column. He thereby obtained a relationship between a and the porosity η, and hence an expression for the multiplication factor m which can be written in the form

$$m^{-1} = 1+\tfrac{3}{4}(1-\eta)\left[1-\left(\frac{8}{1-\eta}-3\right)^{\frac{1}{2}}\right]. \tag{2.3}$$

This requires that the permeability be given by

$$K = K_0/m, \tag{2.4}$$

where K_0 is the value of K in the limit as $\eta \to 1$. According to (2.3), m becomes unbounded as $\eta \to \frac{1}{3}$, and hence we must assume that $\frac{1}{3} < \eta < 1$. Brinkman showed that (2.4) was in qualitative agreement with an experimental relation (Carman–Kozeny) which is now widely accepted, namely

$$K = \frac{d_p^2\,\eta^3}{180(1-\eta)^2}. \tag{2.5}$$

According to Lundgren (1972), Brinkman should have identified v_0 not with the mean filter velocity, but rather with that velocity divided by the porosity. The effect of this change is that in (2.3) one should replace $1-\eta$ by $(1-\eta)/\eta$ so that one then has

$$m^{-1} = 1+\frac{3(1-\eta)}{4\eta}\left[1-\left(\frac{8\eta}{1-\eta}-3\right)^{\frac{1}{2}}\right], \tag{2.6}$$

and this requires that $0{\cdot}6 < \eta < 1$. Since most naturally occurring porous media have porosities less than 0·6 this restriction is indeed restrictive.

Brinkman's reasoning is heuristic. To describe the mean flow past a particular sphere he set up an effective medium, defined by the simplest differential equation that would reduce to Darcy's law for uniform velocity, and to the equation of slow viscous flow for small-scale velocity variations. However, the work of Tam (1969), Lundgren (1972), Childress (1972) and Howells (1974) has left no doubt that the Brinkman model gives the correct first correction to the Stokes drag formula for sparse distributions of spheres. Howells noted that the difference between the distributed resistance of Brinkman's model and the localized resistance of the actual system will not affect, to leading order in c (the mean volume fraction occupied by the spheres), the backflow set up by the fixed array as a reaction to the flow due to a test sphere.

The question of whether one should put the effective viscosity $\tilde{\mu}$ equal to the fluid viscosity μ, or to a viscosity that accounts for the concentration of the particles as Einstein's correction does for dilute suspensions, was answered by Lundgren (1972). Brinkman took $\tilde{\mu} = \mu$, but Lundgren concluded that, if one interpreted $\mathbf{v}$ as the ensemble average of the velocity field and p as the mean static pressure in the fluid, then $\tilde{\mu}/\mu$ was a function, whose value rose slightly above 1 as the porosity η decreased from unity, attained a maximum at about $\eta = 0{\cdot}8$, and decreased rapidly when $\eta < 0{\cdot}7$.

Experimental checks of Brinkman's theory have been indirect and few in number.

Lundgren refers to measurements of flow through cubic arrays of spherical beads on wires by Happel & Epstein (1954) which agree quite well with the Brinkman formula for permeability as a function of porosity.

It was pointed out by Tam (1969) that whenever the spatial length scale is much greater than $1/\alpha_1$ (where α_1 is defined by $\mu/K = \tilde{\mu}\alpha_1^2$) the $\nabla^2\mathbf{v}$ term is negligible in comparison with the term linear in $\mathbf{v}$, and the Brinkman equation reduces to the Darcy equation. Indeed, Levy (1981) has claimed that the Brinkman model really only holds for particles whose size is of order η_1^3 if $\eta_1 \ll 1$ is the distance between two neighbouring particles; for larger particles the fluid filtration is governed by Darcy's law and smaller particles do not influence the flow.

In their discussion of flow through fibrous material, Spielman & Goren (1968, pp. 280, 281) have made several pertinent comments. They write: '... the Brinkman hypothesis implies that the neighbouring fibers damp the ensemble average microscopic flow near the central fiber in precisely the same way that the fibers of the medium damp local flow through the medium when averaged over all conceivable fiber arrangements ... [and so] the validity of Brinkman's hypothesis may be expected to be limited to conditions where the neighbouring fibers are distributed about the central fiber in approximately the same way as they are generally distributed in the medium. The hypothesis therefore breaks down when applied to media of sufficiently low porosity because the effect of many solid boundaries in the immediate proximity to the central object cannot be well described by a simple damping term with spatially and directionally constant damping coefficients. For the case of packed spheres, Brinkman [1947*b*] has shown that allowing a simple spatial variation of the damping coefficient enables extension of the model to low porosities but at the expense of introducing a semi-empirical fitting parameter ... .

'Very near the central fiber where the fluid velocity is small, the damping forces will be negligible when compared with the viscous forces and the Brinkman equation reduces to Stokes' equation of creeping flow. Far from the fiber, where velocity gradients vanish (on the average), [it] reduces to Darcy's equation. Brinkman's equation is seen to be the simplest postulate enabling these two limiting forms. Its validity in the region where both terms are of similar magnitude is open to question.'

3. The Brinkman equation applied to the Rayleigh–Darcy convection problem

It appears that Katto & Masuoka (1967) were unaware of Brinkman's work when they introduced their equation (which is equivalent to Brinkman's with $\tilde{\mu} = \mu$) in an *ad hoc* manner, 'for the sake of convenience'. The Brinkman equation is a convenient linear equation containing a parameter (the permeability K) such that the equation reduces to the Navier–Stokes equation as $K \to \infty$ and to the Darcy equation as $K \to 0$. Katto & Masuoka found that the Darcy equation was applicable if $K/l^2 < 10^{-3}$, where l is the layer depth. Walker & Homsy (1977) calculated the critical Rayleigh number against K/l^2 for the case of conducting no-slip boundaries. They made no effort to justify their equations of motion, other than calling them 'Darcy–Brinkman–Boussinesq equations'.

Rudraiah *et al.* (1980) have adopted without questioning the Katto–Masuoka version of the Brinkman equation, and have applied it to the study of the onset of convection with nonlinear basic temperature profiles. It is pertinent to look at their results for the special case of a linear temperature profile for the situation where both boundaries are rigid and subject to constant heat flux. With the Rayleigh–Bénard number R defined in the usual way for a viscous fluid (see (5.23) below) they calculate the critical value of R to be

$$R_c = 720 + 17{\cdot}14\, l^2/K. \tag{3.1}$$

When $K \to \infty$, this gives $R_c \to 720$, the well-known critical value for the viscous-fluid problem. However, with $K \to 0$ it yields the result $(K/l^2)\, R_c \to 17{\cdot}14$. We recognize that $(K/l^2)\, R$ is identical with the Rayleigh–Darcy number R_m, and we recall that for the case of impermeable boundaries at constant heat flux the critical value R_{mc} is 12. One would expect the value 17·14 to be in error by a few per cent, since Rudraiah *et al.* used a one-term Galerkin approximation in their calculation, but the discrepancy between 17·14 and 12 is far too large to be acceptable. One would expect that the fact that the boundaries are rigid rather than stress-free, so that the non-slip requirement is imposed, would lead to a small correction to the value of R_{mc} if the layer depth is large in comparison with the pore-size, because in this situation the Darcy equation should be applicable everywhere except in a narrow boundary layer near the rigid boundaries. For example, in the heat-transfer experiments of Elder (1967), in which the boundaries were rigid (and at constant temperature in this case), the experimental value of R_{mc} agreed with the theoretical value to well within the 10 % estimated error of measurement, so that the boundary-layer correction could not be larger than about 10 %. We may conclude that it is not always justifiable to use the Brinkman equation within the bulk of a porous medium whose porosity is not close to unity. This leaves still open the question of whether the Brinkman equation is applicable in regions close to the boundaries.

4. Relationship between the Brinkman equation and the Beavers–Joseph boundary condition

Soon we will be making use of an empirical boundary condition proposed by Beavers & Joseph (1967), denoted below by BJ. This condition can be written as

$$\frac{\partial u}{\partial y} = \frac{\alpha}{K^{\frac{1}{2}}}(u - u_m). \tag{4.1}$$

Here it is supposed that there is unidirectional flow in the x-direction, parallel to the plane $y = 0$. The region $y > 0$ is occupied by fluid, and the region $y < 0$ is occupied by a porous medium saturated by that fluid, while u and u_m are the velocity in the fluid and the seepage velocity (mean filter velocity) in the porous medium, respectively. It is understood that in (4.1) u and $\partial u/\partial y$ are evaluated at $y = 0+$, while u_m is evaluated at asome small distance from the plane $y = 0$, so there is a thin layer just inside the medium over which the transition from $u|_{y=0+}$ to u_m takes place (see figure 1 of BJ). The quantity α is a dimensionless quantity, independent of the viscosity of the fluid but depending on the material parameters that characterize the structure of permeable material within the boundary region. In their experiments BJ found that α had the values 0·78, 1·45 and 4·0 for Foametal having average pore sizes 0·06, 0·034 and 0·045 inches respectively, and 0·1 for Aloxite with average pore size 0·013 or 0·027 in. The original experiments of BJ were not conclusive, but more evidence for the correctness of their boundary condition was provided by Beavers, Sparrow & Magnuson (1970) and Beavers, Sparrow & Masha (1974). Some theoretical support for the BJ condition is given by the results of Taylor (1971) and Richardson (1971) based on an analogue model of a porous medium, and by the statistical treatment of Saffman (1971). The latter pointed out that the precise form of the BJ condition was special to the planar geometry in the situation considered by BJ, and in general was not correct to order K. Saffman showed that on the boundary

$$u = \frac{K^{\frac{1}{2}}}{\alpha}\frac{\partial u}{\partial n} + O(K), \tag{4.2}$$

where n refers to the direction normal to the boundary. (In (4.1) u_{m} is $O(K)$ and may be neglected if one wishes.) Jones (1973) assumed that the BJ condition was essentially a relationship involving shear stress rather than just velocity shear. On this view (4.1) would generalize to

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\alpha}{K^{\frac{1}{2}}}(u - u_{\mathrm{m}}), \tag{4.3}$$

where $\mathbf{u} = (u, v, w)$. As far as the author is aware, the formula (4.3) has not been confirmed. The results obtained in the present paper and in Nield (1977) do not depend on the choice between (4.1) and (4.3), and hence we use the simpler version, namely (4.1). (The two-layer convection problem with *conducting* boundaries would provide a test case, because convection would occur at finite rather than zero wavenumber, and the term $\partial v/\partial x$ would no longer be zero.) The BJ condition has been used, with apparent success, by a number of authors in their discussion of porous journal bearings and squeeze films.

Our present interest is centred on the observation by Taylor that the BJ boundary condition can be deduced as a consequence of the use of the Brinkman equation. This idea was developed in detail by Neale & Nader (1974), who showed that in the problem of flow in a channel bounded by a thick porous wall one gets the same solution with the Brinkman equation as one gets with the Darcy equation plus the BJ boundary condition, provided that one identifies α with $(\tilde{\mu}/\mu)^{\frac{1}{2}}$. The successful use of the Brinkman equation here is apparently due to the fact that the velocity within the porous medium is constant except in the region close to the boundary, and thus the Laplacian term is zero except in that region. The lack of success with the Brinkman equation for the Rayleigh–Darcy problem seems to be due to failure to take explicit account of the fact that the neighbourhood of the boundary is a distinguished region.

5. A layered model

We make a fresh start on the convection problem and suppose that we have a porous-medium layer which is sandwiched between two fluid layers, the whole sandwich lying between two rigid horizontal boundaries. For convenience of analysis we apply constant-heat-flux boundary conditions. The problem is thus closely similar to that treated by Nield (1977), so we merely have to modify the analysis presented in that paper.

We choose Cartesian coordinates with the z-axis vertically upwards. We suppose that the porous medium occupies the region $-d_{\mathrm{m}} \leqslant z \leqslant d_{\mathrm{m}}$ and that fluid, identical with that saturating the medium, occupies the regions $d_{\mathrm{m}} \leqslant z \leqslant d_{\mathrm{m}} + d$ and $-(d_{\mathrm{m}} + d) \leqslant z \leqslant -d_{\mathrm{m}}$. We thus have a porous layer of thickness $2d_{\mathrm{m}}$ bounded by fluid layers, each of thickness d. The situation is thus similar to that investigated by Masuoka (1974), but, whereas he treated a thin layer of porous medium between two thick layers of fluid, we are interested in the situation where a thick layer of porous medium lies between very thin layers of fluid, so that d is of the order of magnitude of the particle size.

In the fluid, the governing equations (for steady flow) are

$$\nabla.\mathbf{u} = 0, \tag{5.1}$$

$$\mathbf{u}.\nabla\mathbf{u} = -\frac{1}{\rho_0}\nabla P + \nu\nabla^2\mathbf{u} - g[1-\alpha^*(T-T_0)]\,\mathbf{e}_z, \tag{5.2}$$

$$\mathbf{u}.\nabla T = \kappa\nabla^2 T, \tag{5.3}$$

while in the porous medium we have

$$\nabla.\mathbf{u}_{\mathrm{m}} = 0, \tag{5.4}$$

$$0 = -\frac{1}{\rho_0}\nabla P_{\mathrm{m}} - \frac{\nu}{K}\mathbf{u}_{\mathrm{m}} - g[1-\alpha^*(T_{\mathrm{m}}-T_0)]\,\mathbf{e}_z, \tag{5.5}$$

$$\mathbf{u}_{\mathrm{m}}.\nabla T_{\mathrm{m}} = \kappa_{\mathrm{m}}\nabla^2 T_{\mathrm{m}}. \tag{5.6}$$

Here, $\mathbf{u}$, T and P are the velocity, temperature and pressure in the fluid, ρ_0 is the fluid density at temperature T_0 (a standard temperature), $\nu = \mu/\rho_0$, where μ is the dynamic viscosity, α^* is the volume-expansion coefficient, while $\kappa = k/\rho_0 c$, where k is the thermal conductivity and c the specific heat (at constant pressure) of the fluid. Similarly $\mathbf{u}_{\mathrm{m}}$, T_{m} and P_{m} are the seepage velocity, temperature and pressure in the porous medium, k_{m} the average thermal conductivity of the medium, and K the permeability. We take T_0 as the temperature at $z = 0$, and suppose that the lower and upper boundaries are at temperatures T_{L} and T_{U} respectively. Then, by symmetry, we have $T_0 = \frac{1}{2}(T_{\mathrm{L}}+T_{\mathrm{U}})$.

In the steady state we have the solution

$$\mathbf{u} = 0,\quad T = \tilde{T} \equiv T_{\mathrm{U}} - \beta(z-d_{\mathrm{m}}-d),\quad P = \tilde{P}, \tag{5.7}$$

$$\mathbf{u}_{\mathrm{m}} = 0,\quad T = \tilde{T}_{\mathrm{m}} \equiv T_0 - \beta_{\mathrm{m}} z,\quad P_{\mathrm{m}} = \tilde{P}_{\mathrm{m}}, \tag{5.8}$$

where β, β_{m} are the temperature gradients in the fluid and medium respectively. Continuity of temperature and heat flux across an interface between fluid and medium requires that

$$T_{\mathrm{U}} + \beta d = T_0 - \beta_{\mathrm{m}} d_{\mathrm{m}},\quad k\beta = k_{\mathrm{m}}\beta_{\mathrm{m}}.$$

From these equations we deduce that

$$\beta = \frac{(T_{\mathrm{L}}-T_{\mathrm{U}})\,k_{\mathrm{m}}}{2(k_{\mathrm{m}}d+kd_{\mathrm{m}})},\quad \beta_{\mathrm{m}} = \frac{(T_{\mathrm{L}}-T_{\mathrm{U}})\,k}{2(k_{\mathrm{m}}d+kd_{\mathrm{m}})}. \tag{5.9}$$

We now suppose that the steady-state conduction solution is perturbed, and we define the perturbation variables

$$\theta = T-\tilde{T},\quad p = P-\tilde{P},\quad \theta_{\mathrm{m}} = T_{\mathrm{m}}-\tilde{T}_{\mathrm{m}},\quad p_{\mathrm{m}} = P_{\mathrm{m}}-\tilde{P}_{\mathrm{m}}.$$

The linearized perturbation equations are (5.1), (5.4) together with

$$\frac{1}{\rho_0}\nabla p = \nu\nabla^2\mathbf{u} + \alpha^* g\theta\mathbf{e}_z, \tag{5.10}$$

$$\beta w + \kappa\nabla^2\theta = 0, \tag{5.11}$$

$$\frac{1}{\rho_0}\nabla p_{\mathrm{m}} = -\frac{\nu}{K}\mathbf{u}_{\mathrm{m}} + \alpha^* g\theta_{\mathrm{m}}\mathbf{e}_z, \tag{5.12}$$

$$\beta_{\mathrm{m}} w_{\mathrm{m}} + \kappa_{\mathrm{m}}\nabla^2\theta_{\mathrm{m}} = 0, \tag{5.13}$$

where $\mathbf{u} = (u, v, w)$, $\mathbf{u}_{\mathrm{m}} = (u_{\mathrm{m}}, v_{\mathrm{m}}, w_{\mathrm{m}})$.

We have the following boundary conditions.

At $z = d_m + d$:
$$w = 0, \quad \frac{\partial w}{\partial z} = 0, \quad \frac{\partial \theta}{\partial z} = 0. \tag{5.14a, b, c}$$

At $z = d_m$:
$$w = w_m, \quad \frac{\partial^2 w}{\partial z^2} = \frac{\alpha}{K^{\frac{1}{2}}}\left(\frac{\partial w}{\partial z} - \frac{\partial w_m}{\partial z}\right), \tag{5.14d, e}$$
$$\left(\frac{\partial^2}{\partial z^2} + 3\nabla_2^2\right)\frac{\partial w}{\partial z} = -\frac{1}{K}\frac{\partial w_m}{\partial z} \tag{5.14f}$$
$$\theta = \theta_m, \quad k\frac{\partial \theta}{\partial z} = k_m\frac{\partial \theta_m}{\partial z}. \tag{5.14g, h}$$

At $z = 0$:
$$\frac{\partial w_m}{\partial z} = 0, \quad \frac{\partial \theta_m}{\partial z} = 0. \tag{5.14i, j}$$

Conditions (5.14i, j) hold because of the symmetry of the problem. The remaining conditions (5.14) are as in Nield (1977).

We now put the equations in non-dimensional form by writing
$$\left.\begin{aligned} &\mathbf{u} = \frac{\kappa}{d}\mathbf{u}', \quad \theta = \beta d\theta', \quad (x, y, z - d_m) = d(x', y', z'), \quad p = \frac{\mu\kappa}{d^2}p'; \\ &\mathbf{u}_m = \frac{\kappa_m}{d_m}\mathbf{u}_m, \quad \theta_m = \beta_m d_m \theta'_m, \quad (x_m, y_m, z_m) = d_m(x'_m, y'_m, z'_m), \quad p_m = \frac{\mu\kappa_m}{K}p'_m. \end{aligned}\right\} \tag{5.15}$$

Substituting in the various equations, and dropping primes, we have the following.

For $0 \leqslant z \leqslant 1$:
$$\nabla \cdot \mathbf{u} = 0, \tag{5.16}$$
$$\nabla p = \nabla^2\mathbf{u} + R\theta\mathbf{e}_z, \tag{5.17}$$
$$w + \nabla^2\theta = 0. \tag{5.18}$$

For $0 \leqslant z_m \leqslant 1$;
$$\nabla_m \cdot \mathbf{u}_m = 0, \tag{5.19}$$
$$\nabla_m p_m = -\mathbf{u}_m + R_m\theta_m\mathbf{e}_{zm}, \tag{5.20}$$
$$w_m + \nabla_m^2\theta_m = 0. \tag{5.21}$$

The boundary conditions become as follows.

On $z = 1$:
$$w = 0, \quad \frac{\partial w}{\partial z} = 0, \quad \frac{\partial \theta}{\partial z} = 0. \tag{5.22a, b, c}$$

On $z = 0$ or $z_m = 1$:
$$\hat{T}w = w_m, \quad \hat{T}\hat{d}\left[\frac{\partial w}{\partial z} - \frac{\delta\hat{d}}{\alpha}\frac{\partial^2 w}{\partial z^2}\right] = \frac{\partial w_m}{\partial z_m}, \tag{5.22d, e}$$
$$\hat{T}\delta^2\hat{d}^3\left(\frac{\partial^3 w}{\partial z^3} + 3\nabla_2^2\frac{\partial w}{\partial z}\right) = -\frac{\partial w_m}{\partial z_m}, \tag{5.22f}$$
$$\theta = \hat{T}\theta_m, \quad \frac{\partial \theta}{\partial z} = \frac{\partial \theta_m}{\partial z_m}, \tag{5.22g, h}$$

On $z_m = 0$:
$$\frac{\partial w_m}{\partial z_m} = 0, \quad \frac{\partial \theta_m}{\partial z_m} = 0. \tag{5.22i, j}$$

Here

$$R = \frac{g\alpha^*\beta d^4}{\kappa\nu}, \quad R_m = \frac{g\alpha^*\beta_m K d_m^2}{\kappa_m \nu}, \tag{5.23}$$

$$\delta = \frac{K^{\frac{1}{2}}}{d_m}, \quad \hat{d} = \frac{d_m}{d}, \quad \hat{k} = \frac{k_m}{k}, \quad \hat{T} = \frac{\beta_m d_m}{\beta d} = \frac{\hat{d}}{\hat{k}}.$$

From (5.16), (5.17), (5.19), (5.20) we derive

$$\nabla^4 w + R\nabla_2^2 \theta = 0, \tag{5.24}$$

$$\nabla_m^2 w_m - R_m \nabla_{2m}^2 \theta_m = 0. \tag{5.25}$$

We now solve (5.18), (5.21), (5.24), (5.25) subject to (5.22). We proceeed as in Nield (1977), making the usual normal-mode expansion, expanding in terms of a^2 (where a is the small horizontal wavenumber for the fluid region) and solving the order-a^0 and order-a^2 equations. After routine algebra we end up with the criterion for the onset of convection in the form

$$\begin{aligned}&[(8+18\Delta)\hat{T} + (15+45\Delta)\hat{T}^2]R \\ &\quad + [120(1+\Delta)\hat{d}^2 + 180\hat{d} + 60\hat{d}/\hat{T} + \delta^{-2}\{(30+120\Delta)\hat{d}^{-1} + (15+45\Delta)\hat{d}^{-1}\hat{T}^{-1}\}]R_m \\ &\qquad = 360(1+\Delta)(\hat{T}+\hat{d}^2),\end{aligned} \tag{5.26}$$

where $\Delta = \delta\hat{d}/\alpha$.

We can check this result for two special cases.

(i) For $\hat{d} \to 0$, we find that $R = 720/2^4$, as expected for a layer of viscous fluid, of depth $2d$, between two rigid boundaries.

(iii) For $\hat{d} \to \infty$, we find that $R_m = 12/2^2$, as expected for a layer of porous medium, of depth $2d_m$, between two impermeable boundaries.

We are interested in the case where $\hat{d}$ is large, so that a thick layer of porous medium lies between two thin fluid layers. We define R^* as the Rayleigh number appropriate for the case where the whole space is occupied by porous medium. i.e. R^* is the Rayleigh–Darcy number for a layer of depth $H = 2d_m + 2d$, so that

$$R^* = \frac{g\alpha^*(T_L - T_U) K H}{\nu \kappa_m}. \tag{5.27}$$

Then R and R_m can be expressed in terms of R^*. We have

$$R_m = \tfrac{1}{4}R^*(1+\hat{d}^{-1})^{-1}(1+\hat{k}\hat{d}^{-1})^{-1}, \tag{5.28}$$

$$R = \tfrac{1}{4}R^*\hat{k}^2\hat{d}^{-4}\delta^{-2}(1+\hat{d}^{-1})^{-1}(1+\hat{k}\hat{d}^{-1})^{-1}. \tag{5.29}$$

The instability criterion can now be written

$$R^* = \frac{12(1+\Delta)(1+\hat{d}^{-1})(1+\hat{k}\hat{d}^{-1})(1+\hat{k}^{-1}\hat{d}^{-1})}{1+\Delta+\dfrac{3}{2\hat{d}}+\dfrac{\hat{k}}{2\hat{d}^2}+\dfrac{1}{\delta^2}\left[\dfrac{(4+9\Delta)\hat{k}}{60\hat{d}^3}+\dfrac{(1+3\Delta)}{8}\left(\dfrac{\hat{k}}{\hat{d}^4}+\dfrac{1}{\hat{d}^4}\right)+\dfrac{1+4\Delta}{4\hat{d}^3}\right]}. \tag{5.30}$$

We compare this with the result $R^* = 12$ which is obtained in the absence of the fluid layers ($\hat{d} \to \infty$).

We note that δ will normally be a small quantity. For example, if we adopt the Carman–Kozeny equation

$$K = \frac{\eta^3 d_p^2}{180(1-\eta)^2}, \tag{5.31}$$

where η is the porosity and d_p a mean particle diameter, and if we suppose that $\eta \approx 0{\cdot}5$, then $K \approx \frac{1}{400}d_p^2$ and so $\delta^2 \approx \frac{1}{400}(d_p/d_m)^2$. If we now consider the case where the fluid layers are of depth d_p, then $\delta^2 \approx \frac{1}{400}\hat{d}^{-2}$ and $\Delta \approx (20\alpha)^{-1}$. Then, to first order in $\hat{d}^{-1}$, we have

$$\tfrac{1}{12}R^* = 1+\left[\hat{k}+1+\hat{k}^{-1}-\frac{100+400\Delta}{1+\Delta}\right]\hat{d}^{-1}. \tag{5.32}$$

The coefficient of $\hat{d}^{-1}$ may be positive or negative, depending on the magnitudes of k_m/k and α. It will be negative unless $\hat{k}$ differs greatly in magnitude from unity. The restricting effect of the no-slip condition, which on its own would lead to an increased critical Rayleigh number, is opposed by the freeing effect due to the fact that there is less viscous dissipation in the fluid than in an equal volume of porous medium.

More generally, we may set $d = \lambda K^{\frac{1}{2}}$, where λ is a numerical parameters which can be chosen to fit experimental results (which are lacking, at present). Then $\hat{d}^{-1} = \lambda\delta$ and, to order δ,

$$\tfrac{1}{12}R^* = 1+\left\{\lambda(\hat{k}+1+\hat{k}^{-1})-\frac{6\lambda+(1+4\Delta)\lambda^3}{4+4\Delta}\right\}\delta, \tag{5.33}$$

where $\Delta = (\lambda\alpha)^{-1}$.

6. Conclusions and discussion

We have shown that although the Brinkman equation is useful in the treatment of flow past a very sparse collection of obstacles, and for flows in porous media where the velocity is constant except in regions near boundaries, it is not generally applicable to flow in porous media. We have described an alternative method of dealing with boundary layers, involving a multilayered model. For the case of constant-heat-flux boundaries, the critical value of the Rayleigh–Darcy number R^* is given by (5.30). This contains the layer depth ratio $\hat{d}$ which can be varied as desired. It is expected that for the case of other thermal boundary conditions the variation of R^* with $\hat{d}$, $\hat{k}$, Δ and δ will be qualitatively similar to that expressed by (5.30). In most practical situations the boundary-layer correction arising from the no-slip requirement will be small.

We have been motivated mainly by the desire to deal with the restrictive effect of the rigid boundary and the associated increase in permeability near that boundary. Also associated with this increase in permeability there will be a change in the thermal conductivity. This has been taken account of, and the effect depends on the value of $\hat{k}$.

In our model the fluid layer is meant to represent a geometrical void, an idealization modelling the reduced density of particles near a boundary. Ideally, we would like to account for variations in the detailed statistics of small ensembles of particles, but, since we do not have a practical method for doing so, we have effectively slashed through the tangle, following the example of Alexander the Great in his treatment of the Gordian-knot problem. Experimental data on the particular aspect of the convection problem which we have discussed is lacking at present, but when some is available it may well be appropriate to consider a less crude approach. A possible step in that direction would be to use a Brinkman-type equation in the boundary region.

In this paper we have been largely concerned with a convection problem, but our discussion has implications for other problems involving fluid flow through a porous medium. One such problem has been considered by Nield (1983).

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D. D. Joseph, T. S. Lundgren and their colleagues in the Department of Aerospace Engineering and Mechanics at the University of Minnesota.

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